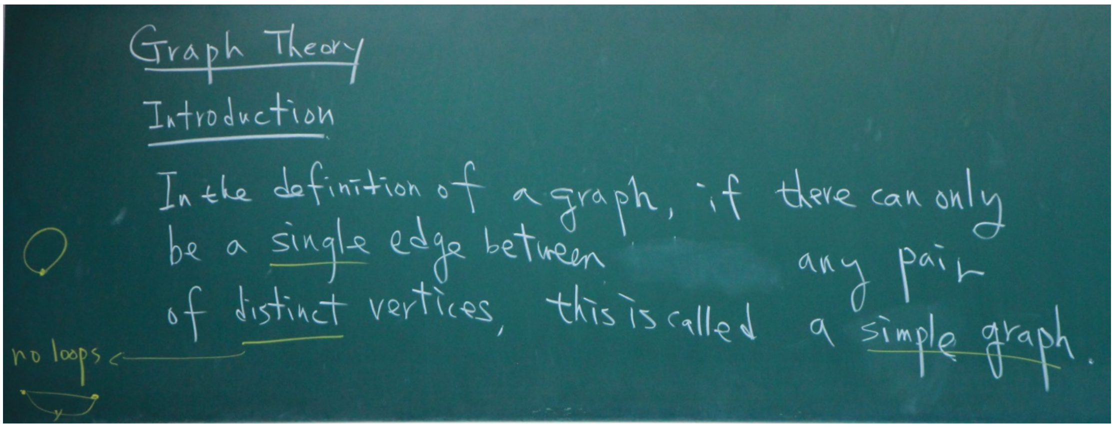
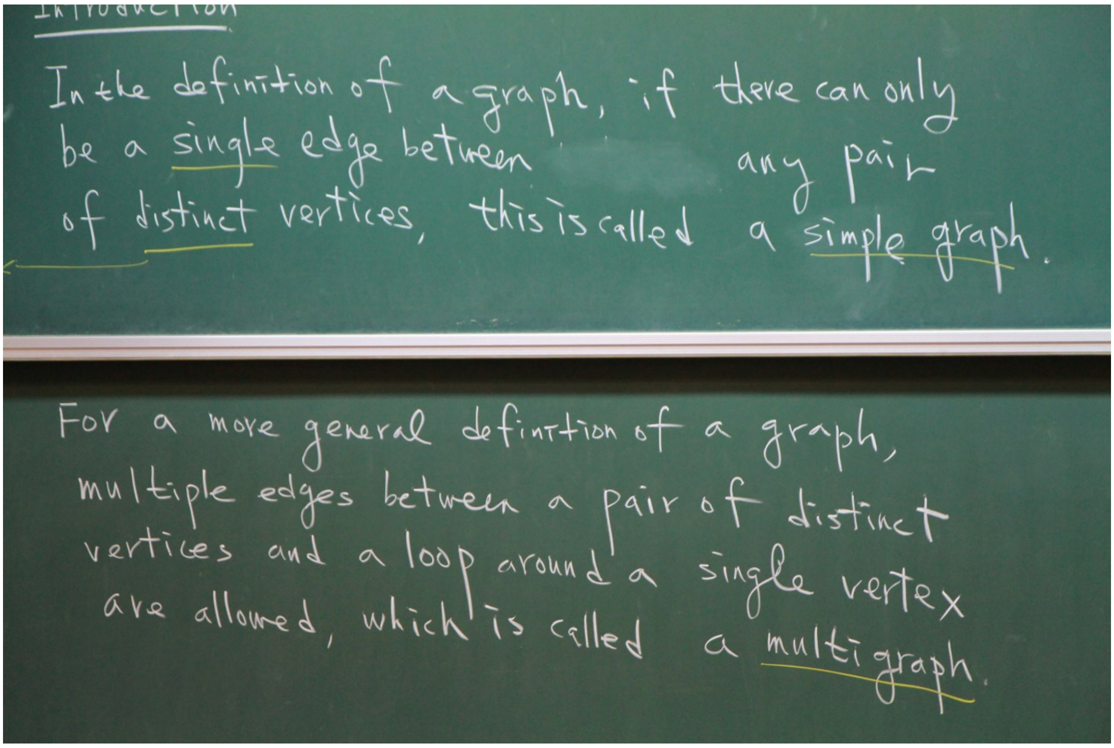
【10920 趙啟超教授離散數學 / 第 19 堂版書】





Graph Theory

Introduction

In the definition of a graph, if there can only be a single edge between any pair of distinct vertices, this is called a simple graph.

no loops

For a more general definition of a graph, multiple edges between a pair of distinct vertices and a loop around a single vertex are allowed, which is called a multigraph.

Vertex Degree

Def For each vertex v of an undirected graph G, the degree of v, $\deg(v)$, is the number of edges that are incident with v.

Remark The definition holds for an undirected simple graph or multigraph. A loop at a vertex v contributes

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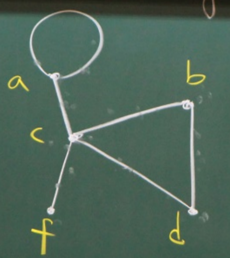

$\deg(a) = 3$
$\deg(b) = 2$
$\deg(c) = 4$
$\deg(d) = 2$
$\deg(e) = 0$
$\deg(f) = 1$

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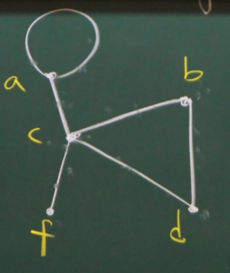

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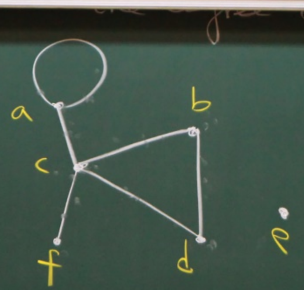

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Theorem $$\sum_{v\in V} \deg(v) = 2|E|.$$

Proof Each edge contributes exactly 2 to the sum of the degrees of all the vertices.

Let $V_o = \{v \in V : \deg(v) \text{ is odd}\}$

and $V_e = \{v \in V : \deg(v) \text{ is even}\}$

Then V_o and V_e form a partition of V.
We have

$$\sum_{v \in V} \deg(v) = \underbrace{\sum_{v \in V_o} \deg(v)}_{\text{even}} + \underbrace{\sum_{v \in V_e} \deg(v)}_{\text{even}}$$

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Property (Handshaking Lemma)

$|V_o|$ is even.

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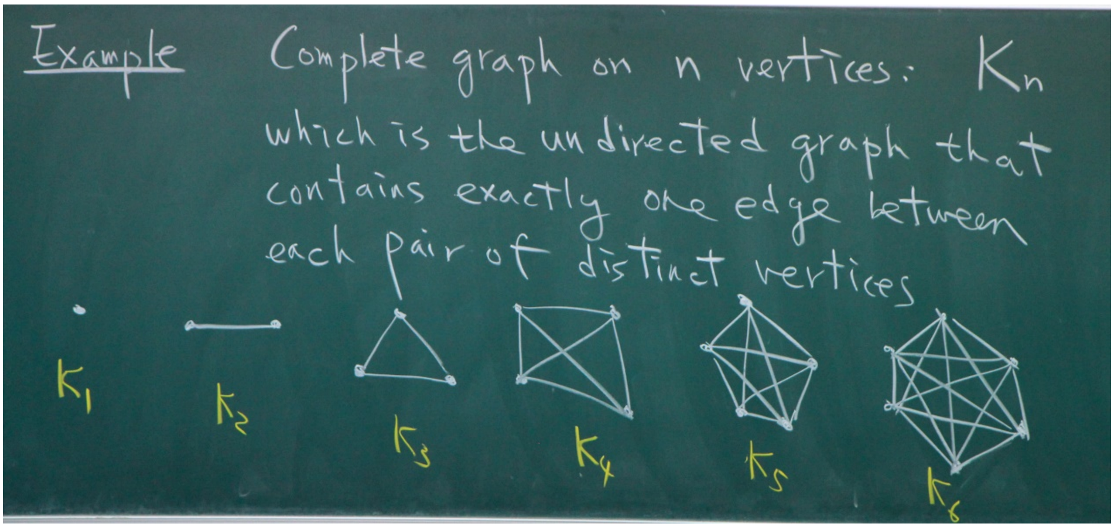
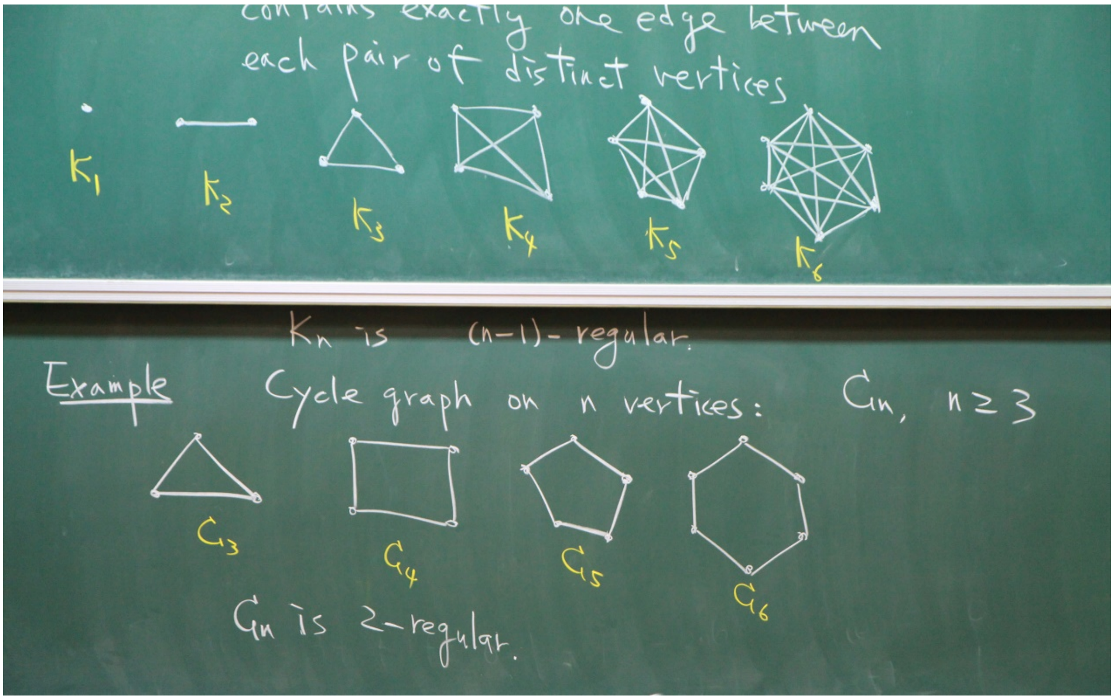


Example
Complete graph on n vertices: Kn
which is the undirected graph that
contains exactly one edge between
each pair of distinct vertices
K1
K2
K3
K4
K5
K6



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Kn is (n-1)-regular.
Example
Cycle graph on n vertices: Cn, n ≥ 3
C3
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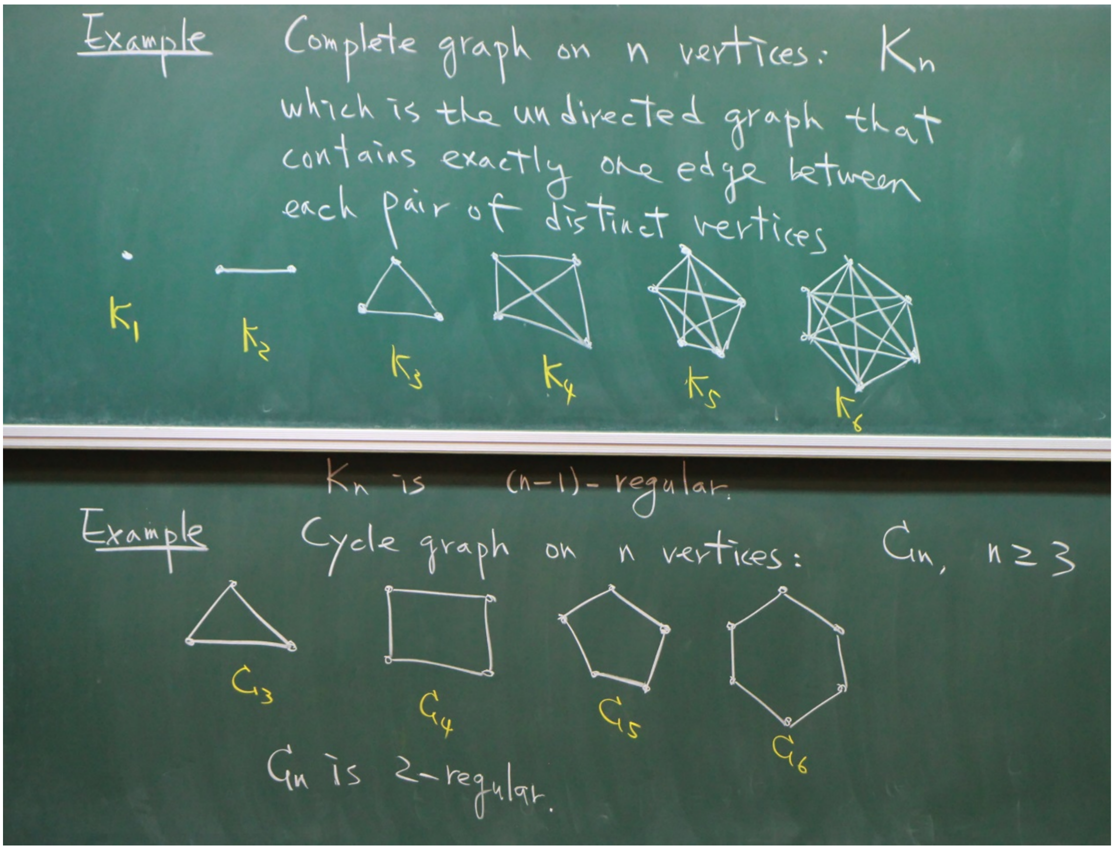
Cn is 2-regular.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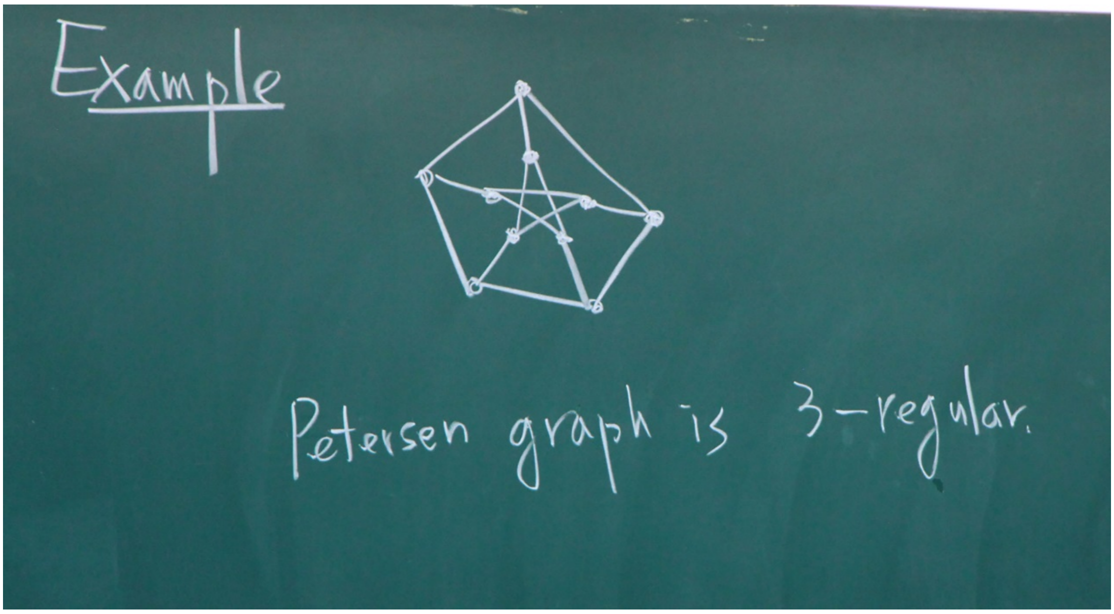
Example
Petersen graph is 3-regular.

Walk, Trail, and Path

Def A walk in an undirected graph G is a finite alternating sequences

$$v_0, e_1, v_1, e_2, v_2, e_3, \cdots, e_{n-1}, v_{n-1}, e_n, v_n$$

of vertices and edges from G, starting at vertex v_0 and ending at vertex v_n, and involving the n edges $e_i = \{v_{i-1}, v_i\}$, $i=1,2,\cdots,n$,

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Then *length* of the walk is n, the number of edges.

A walk with $v_0 = v_n$ is called a *closed* walk.

Otherwise the walk is called *open*.

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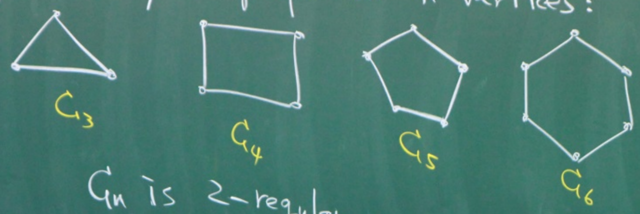
K_n is $(n-1)$-regular.

Example Cycle graph on n vertices: C_n, $n \geq 3$



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Repeated vertices	Repeated edges	Open/closed	Name
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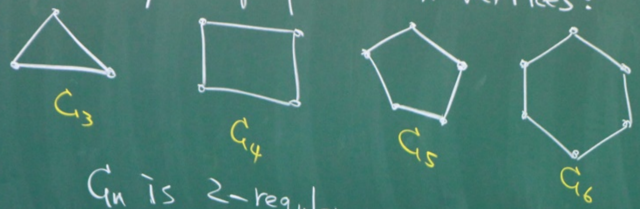

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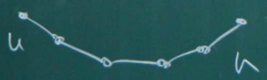
Define $u \sim v$ whenever there is a path between vertices u and v of G.



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So V is partitioned into disjoint equivalence classes. Two vertices are in the same equivalence class iff there is a path between them.

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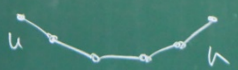

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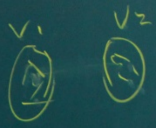

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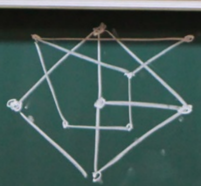

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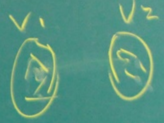

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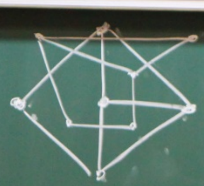

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